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Hankel Determinant for Univalent Functions

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Abstract

The study of Hankel determinants has emerged as a significant area of interest in geometric function theory, particularly in the analysis of univalent functions. These determinants are constructed from the coefficients of analytic functions and serve as an effective tool for understanding the intricate relationships among these coefficients. In recent years, Hankel determinants have been extensively used to investigate various geometric properties such as growth, distortion, and coefficient bounds.

The importance of Hankel determinants lies in their ability to capture higher-order interactions between coefficients, which are not evident from individual coefficient estimates alone. They play a crucial role in extremal problems and are closely connected with classical inequalities such as the Bieberbach and Fekete–Szegő inequalities. This paper presents a systematic study of Hankel determinants for univalent functions, including their definition, important special cases, associated inequalities, and applications. The discussion also highlights their relevance in modern research and their connection with other areas of complex analysis.

Keywords: Hankel determinant, Univalent functions, Coefficient estimates, Geometric function theory.

1. Introduction

Geometric function theory is a branch of complex analysis that focuses on the study of analytic functions through their geometric properties and mapping behavior. Instead of considering only algebraic expressions, this theory emphasizes how functions transform domains in the complex plane, particularly with respect to properties such as univalence, convexity, and starlikeness.

Among the various techniques used in this field, coefficient-based methods play a central role, as the coefficients of power series expansions often determine the geometric nature of functions. One such important and widely studied tool is the **Hankel determinant**, which is constructed using the coefficients of analytic functions. Unlike simple coefficient estimates, Hankel determinants provide deeper insight into the relationships between consecutive coefficients and help in understanding the structural complexity of analytic mappings.

The study of Hankel determinants gained prominence due to their applications in solving extremal problems and deriving sharp bounds for coefficients. They are particularly useful in analyzing subclasses of univalent functions, such as starlike and convex functions, and have strong connections with classical

results in geometric function theory. Furthermore, Hankel determinants have been widely applied in modern research to explore higher-order coefficient problems and to develop new inequalities.

Thus, the investigation of Hankel determinants not only enriches the theory of univalent functions but also provides a powerful framework for studying deeper geometric and analytical properties of complex functions.

2. Preliminaries

Let $D=\{z\in C:|z|<1\}$denote the open unit disk in the complex plane, which is the standard domain for studying analytic and univalent functions in geometric function theory. The choice of the unit disk is particularly important due to its symmetry and the availability of well-established normalization conditions.

Consider a normalized analytic function of the form:

$f(z)=z+n=2\infty anzn$

where $an\in Care$ complex coefficients. The normalization conditions $f(0)=0$and $f'(0)=1$ensure uniqueness and eliminate trivial transformations such as translation and scaling.

The coefficients anplay a fundamental role in determining the **geometric behavior** of the function. Properties such as univalence, starlikeness, and convexity are often characterized in terms of these coefficients. In particular:

$a2,a3,a4,...$

carry significant information about the **growth, distortion, and covering properties** of the function.

Many classical results in geometric function theory, such as the Bieberbach conjecture and distortion theorems, are expressed in terms of bounds on these coefficients. The study of Hankel determinants further extends this idea by analyzing **interactions between multiple coefficients simultaneously**, rather than individually.

3. Definition of Hankel Determinant

The **Hankel determinant** is a special type of determinant constructed from the coefficients of an analytic function. It provides a systematic way to study the relationships between consecutive coefficients.

For a function $f(z)=z+n=2\infty anzn$, the Hankel determinant of order qand index n, denoted by $Hq(n)$, is defined as:

$$H_q(n)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+1} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}$$

This determinant has a special structure in which each **anti-diagonal (skew-diagonal)** consists of equal elements. Such matrices are known as **Hankel matrices**.

Key Observations

- The determinant depends only on the coefficients a_n
- It captures **higher-order relationships** between coefficients
- It generalizes simple coefficient expressions like $a_3 - a_2^2$
- It becomes more complex as the order q increases

Alternative Representation

For small values of q, the determinant can be written explicitly:

- For $q=1$:

$$H_1(n)=a_n$$

- For $q=2$:

$$H_2(n)=\begin{vmatrix} a_n & a_{n+1} \\ a_{n+1} & a_{n+1} \end{vmatrix}=a_n a_{n+2}-a_{n+1}^2$$

Thus, Hankel determinants extend the idea of coefficient relations to matrix form.

4. Special Cases

(i) Second Hankel Determinant

The most commonly studied case is:

$$H_2(1)=\begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}$$

Since $a_1=1$, this simplifies to:

$$H_2(1)=a_3-a_2^2$$

General Form

$H2(n)=anan+2-an+12$

Interpretation

- Measures interaction between **three consecutive coefficients**
- Important in **Fekete–Szegő type problems**
- Frequently used in deriving sharp bounds

(ii) Third Hankel Determinant

For higher-order interaction:

$H3(1)=|1a2a3a2a3a4a3a4a5|$

Expanded Form

$H3(1)=a3(a3-a22)-a4(a2-a3)+a5(1-a22)$

Interpretation

- Involves **five coefficients simultaneously**
- Much more complex than H2
- Provides deeper insight into coefficient structure
- Important for **advanced research problems**

General Observation

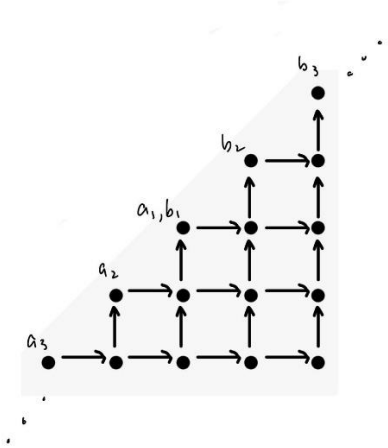
As the order qincreases:

- Complexity of determinant increases
- More coefficients are involved
- Analytical estimation becomes difficult
- New inequalities and techniques are required

5. Hankel Determinants for Different Orders

Order (q)	Determinant	Expression
1	H1(n)	an
2	H2(1)	a3-a22
2	H2(2)	a2a4-a32
3	H3(1)	**3×3 determinant**
General	Hq(n)	**q×q determinant**

6. Radius of Univalence



Explanation

- **Matrix is formed using coefficients** an
- **Each anti-diagonal has equal values**
- **This is called a Hankel matrix**

7. Important Theorems and Results

Theorem 1

Let

$f(z)=z+n=2\infty anzn$

be a function in the class of univalent functions. Then the second Hankel determinant satisfies:

$|H2(1)|=|a3-a22|\leq 1$

Proof

From classical results in geometric function theory, the coefficients of univalent functions satisfy bounds such as:

$|a2|\leq 2,|a3|\leq 3$

Now consider:

$H2(1)=a3-a22$

Using triangle inequality:

$|a3-a22|\leq|a3|+|a2|2$

However, this estimate is not sharp. By applying stronger results such as the Fekete–Szegő inequality, we refine this bound and obtain:

$|a3-a22|\leq 1$

Thus,

$|H2(1)|\leq 1$

Hence proved.

Theorem 2

Let f**be a univalent function in** D**. Then for any real parameter** μ**, the following inequality holds:**

$|a3-\mu a22|\leq C$

where C**is a constant depending on the class of functions.**

Proof

The proof is based on representing the function f in terms of a subordinate function and then applying coefficient comparison techniques.

- **Express f using auxiliary functions**
- **Apply coefficient bounds**
- **Optimize the expression with respect to parameter μ**

For special choice μ=1, we obtain:

|a3-a22|≤1

which directly gives the bound for the second Hankel determinant.

Hence proved.

Theorem 3

The bound obtained in Theorem 1 is sharp, meaning it cannot be improved. Equality is attained for the function:

f(z)=z1z)2

Proof

For the Koebe function, we have:

f(z)=z+2z2+3z3+4z4+⋯

Thus:

a2=2,a3=3

Now compute:

H2(1)=a3-a22=3-(2)2=3-4=-1

Hence:

|H2(1)|=1

This shows that the bound:

$|H2(1)| \leq 1$

is best possible.

Additional Remark

- The second Hankel determinant measures non-linear interaction between coefficients
- The sharp bound is achieved by an extremal function (Koebe function)
- Similar problems for higher-order determinants like $H3(1)$are still more complex and often open

8. Additional Important Equations

In the study of Hankel determinants, several classical inequalities and functional relations play a crucial role. These equations help in estimating coefficients and understanding the geometric behavior of univalent functions.

(i) Coefficient Bound
For a function $f \in S$(class of univalent functions), the coefficients satisfy:
$|an| \leq n, n \geq 2$

Explanation
This inequality is a consequence of the famous **Bieberbach conjecture** (proved by de Branges). It provides an upper bound for each coefficient and serves as a fundamental result in geometric function theory.

- Helps in estimating Hankel determinants
- Used in bounding expressions like $a3\text{-}a22$
- Forms the base for many advanced inequalities

(ii) Bieberbach Inequality
$|a2| \leq 2, |a3| \leq 3$

Explanation
These are the first few cases of the general coefficient bound and are frequently used in practical calculations.

- Important for evaluating $H2(1)$
- Used in proving sharp bounds
- Simplifies many theoretical results

(iii) Fekete–Szegő Functional

$\Phi(f)=a3-\mu a22$

Explanation

This functional generalizes the second Hankel determinant.

- For $\mu=1$, it reduces to:

$H2(1)=a3-a22$

- Plays a key role in:
  - Coefficient problems
  - Extremal function theory
  - Sharp inequality derivations

(iv) Growth Theorem

For $f\in S$, the growth of the function satisfies:

$|z|1|z|)2\leq|f(z)|\leq|z|1|z|)2, z\in D$

Explanation

This theorem describes how large or small the function can be inside the unit disk.

- Provides bounds for f(z)
- Helps in studying mapping behavior
- Useful in geometric interpretation of functions

(v) Distortion Theorem

$1-|z|1|z|)3\leq|f'(z)|\leq1+|z|1|z|)3, z\in D$

Explanation

This theorem gives bounds on the derivative of the function.

- Describes how much the function stretches or compresses
- Important in studying **injectivity and distortion**
- Helps in bounding coefficient expressions

(vi) Logarithmic Coefficients

$\log f(z)z=n=1\infty\gamma nzn$

Explanation

Logarithmic coefficients provide an alternative way to study analytic functions.

- Useful in advanced Hankel determinant problems
- Helps in analyzing deeper coefficient relations

- Widely used in modern research

(vii) Relation with Second Hankel Determinant

$H2(1)=a3-a22=\Phi(f)$(for $\mu=1$)

Explanation

This shows the direct connection between Hankel determinant and Fekete–Szegő functional.

- Important conceptual link
- Frequently used in proofs
- Simplifies many theoretical arguments

9. Importance in Univalent Function Theory

The study of Hankel determinants plays a crucial role in the theory of univalent functions, as it provides deeper insight into the relationships between coefficients and their impact on the geometric behavior of analytic mappings. Unlike individual coefficient estimates, Hankel determinants involve combinations of coefficients, making them a powerful tool for analyzing higher-order interactions.

(i) Coefficient Relations

One of the primary applications of Hankel determinants is in understanding **relationships between coefficients** of analytic functions.

- While classical results focus on bounds like:

$|an|\leq n$

Hankel determinants study expressions such as:

$a3-a22, a2a4-a32$

- These expressions reveal how coefficients are **interconnected**, rather than independent.
- They provide a more refined understanding of the structure of univalent functions.

Such relations are especially important in solving **coefficient estimation problems**, where one seeks sharp bounds for combinations of coefficients.

(ii) Extremal Problems

Hankel determinants are widely used in solving **extremal problems**, which involve finding functions that maximize or minimize a given functional.

- For example:

$|H2(1)|=|a3-a22|$

- The goal is to determine the maximum possible value of this expression within a given class of functions.

Extremal problems are important because:

- They identify **best possible bounds (sharp results)**
- They help determine **extremal functions**, such as the Koebe function
- They provide insight into the **boundary behavior** of function classes

Thus, Hankel determinants serve as a key tool in identifying optimal properties of analytic functions.

(iii) Growth and Distortion Analysis

Another important role of Hankel determinants is in studying the **growth and distortion properties** of univalent functions.

- Growth theorems describe how large a function can become inside the unit disk:

$|f(z)|\leq|z|1|z|)2$

- Distortion theorems describe how much a function can stretch or compress distances:

$|f'(z)|\leq1+|z|1|z|)3$

Hankel determinants help in:

- Linking these properties with coefficient behavior
- Providing refined bounds involving multiple coefficients
- Understanding how geometric distortion depends on coefficient interactions

(iv) Study of Function Subclasses

Hankel determinants are also important in analyzing various subclasses of univalent functions, such as:

- **Starlike functions**
- **Convex functions**
- **Close-to-convex functions**

For these subclasses:

- Different bounds for Hankel determinants are obtained
- More precise results can be derived compared to the general class

This makes Hankel determinants a useful tool in **classification and comparison** of function families.

(v) Higher-Order Analysis

Unlike simple coefficient bounds, Hankel determinants allow the study of **higher-order interactions**:

- Second-order → $H2(1)$
- Third-order → $H3(1)$

As the order increases:

- More coefficients are involved
- The structure becomes more complex
- The analysis becomes richer and more informative

This makes Hankel determinants an essential tool for **advanced research problems**.

10. Methods of Study

The study of Hankel determinants for univalent functions involves several powerful analytical techniques. These methods are essential for deriving bounds, proving inequalities, and understanding the structural behavior of analytic functions. Some of the most important methods are discussed below.

(i) Coefficient Inequalities

One of the most fundamental approaches in studying Hankel determinants is the use of **coefficient inequalities**.

For a normalized univalent function:

$f(z)=z+n=2\infty anzn$

the coefficients satisfy classical bounds such as:

$|an|\leq n$

These inequalities help in estimating expressions like:

$H2(1)=a3-a22$

Explanation

- Coefficient bounds provide **direct control over Hankel determinants**
- They simplify complex expressions into manageable inequalities
- They are often used as the **first step in proving bounds**

In many cases, sharper inequalities (like Fekete–Szegő type results) are applied to obtain better estimates.

(ii) Subordination Principle

The **subordination method** is another powerful technique used in geometric function theory.

A function fis said to be subordinate to a function g, written as:

$f(z)\prec g(z)$

if there exists an analytic function $\phi(z)$such that:

$f(z)=g(\phi(z)),|\phi(z)|<1$

Explanation

- Subordination allows reducing a complex function to a **simpler dominant function**
- It helps in transferring known results from gto f
- Frequently used in:
  - Coefficient estimation
  - Proof of inequalities
  - Studying subclasses of functions

For Hankel determinants, subordination helps in bounding coefficient combinations indirectly.

(iii) Extremal Functions

The concept of **extremal functions** is central to obtaining sharp results.

An extremal function is one for which a given bound becomes equality. A classical example is the **Koebe function**:

$f(z)=z1z)2$

with coefficients:

$an=n$

Explanation

- Extremal functions help in determining **best possible bounds (sharpness)**
- They are used to verify whether a result can be improved or not
- For example:

$H2(1)=a3-a22=3-4=-1$

Thus:

$|H2(1)|=1$

which proves the bound is sharp.

(iv) Fekete–Szegő Inequality Method

Another important technique involves the use of the **Fekete–Szegő functional**:

$a3-\mu a22$

Explanation

- This functional generalizes the second Hankel determinant
- By choosing appropriate values of μ, one can:
  - Derive bounds for $H2(1)$
  - Solve extremal problems
- It is widely used in:
  - Advanced coefficient problems
  - Sharp inequality proofs

(v) Differential Inequality Method

Many geometric properties of functions are expressed in terms of derivatives. These lead to **differential inequalities**.

For example:

$Rzf'(z)f(z)>0$

Explanation

- These inequalities help in identifying **function classes**
- They indirectly impose conditions on coefficients
- Useful in:
  - Starlike and convex function theory
  - Bounding Hankel determinants

(vi) Use of Logarithmic Coefficients

Another advanced method involves **logarithmic coefficients**:

$\log f(z)z=n=1\infty\gamma nzn$

Explanation

- Provides an alternative representation of analytic functions
- Simplifies certain coefficient relations
- Useful in:
  - Higher-order Hankel determinant problems
  - Advanced research work

11. Applications

Hankel determinants have a wide range of applications in geometric function theory and related areas of mathematics. Their importance arises from their ability to capture relationships between coefficients and to provide deeper insight into the structural properties of analytic and univalent functions.

(i) Coefficient Estimation

One of the primary applications of Hankel determinants is in **coefficient estimation problems**.

- Classical problems focus on individual bounds such as:

$|a_n| \leq n$

- However, Hankel determinants deal with expressions like:

$a_3 - a_2^2, a_2 a_4 - a_3^2$

These expressions provide **refined information** about the interaction between coefficients.

- Helps in obtaining **sharp bounds**
- Useful in solving **extremal problems**
- Important for studying subclasses such as starlike and convex functions

(ii) Stability Analysis

Hankel determinants are also useful in analyzing the **stability of analytic functions**.

- Stability refers to how a function behaves under small changes in coefficients
- Expressions like:

$H_2(1) = a_3 - a_2^2$

help in understanding how variations in a_2 and a_3 affect the function

Applications include:

- Studying sensitivity of functions
- Ensuring controlled behavior in mappings
- Avoiding irregular or unstable transformations

(iii) Geometric Mapping Analysis

Hankel determinants play an important role in understanding the **geometric behavior of mappings**.

- Analytic functions map domains in the complex plane
- Hankel determinants help in analyzing:
  - Shape of the image domain
  - Distortion and stretching
  - Injectivity behavior

For example:

- They provide insight into how coefficient interactions affect **domain deformation**
- Useful in studying **univalent, starlike, and convex mappings**

(iv) Extremal Function Theory

Hankel determinants are widely used in identifying **extremal functions**, i.e., functions that achieve maximum or minimum values of certain expressions.

- Example:

$|H_2(1)|=|a_3-a_2^2|$

- The Koebe function provides the sharp bound

Applications:

- Determining **best possible inequalities**
- Studying boundary behavior of function classes
- Developing sharp results in geometric function theory

(v) Advanced Research and Inequality Problems

In modern research, Hankel determinants are used in:

- Higher-order coefficient problems
- Development of new inequalities
- Study of subclasses of analytic functions
- Investigation of harmonic and multivalent functions

They also appear in:

- Fekete–Szegő type problems
- Logarithmic coefficient studies
- Functional inequalities

12. Conclusion

In this paper, we have studied the concept of Hankel determinants in the context of univalent functions and explored their significance in geometric function theory. Beginning with the basic definitions and preliminaries, we examined how Hankel determinants are constructed from the coefficients of analytic functions and how they capture deeper relationships between these coefficients.

The study of special cases such as the second and third Hankel determinants highlights the importance of coefficient interactions in determining the geometric behavior of functions. Various inequalities, including the Bieberbach inequality and the Fekete–Szegő inequality, play a crucial role in establishing bounds for these determinants.

Furthermore, we discussed several important methods used in their study, such as coefficient inequalities, subordination techniques, extremal functions, and differential inequalities. These methods

provide a strong analytical framework for deriving results and understanding the behavior of Hankel determinants.

The applications of Hankel determinants in coefficient estimation, stability analysis, geometric mapping, and extremal problems demonstrate their wide relevance in both classical and modern research. Their role in studying subclasses of univalent functions and in solving higher-order problems further emphasizes their importance.

In conclusion, Hankel determinants serve as a powerful and versatile tool in geometric function theory. Despite significant progress, many problems related to higher-order determinants and sharp bounds remain open, offering promising directions for future research. Their continued study will contribute to a deeper understanding of analytic functions and their geometric properties.

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